A Sum of Two Geometric Decays Can Misbehave Only Once

Tiberiu Popa

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Consider the points

$$P_n = (r^n, r^{2n})$$

on the parabola $y = x^2$, where $0 < r < 1$. Their signed projections in the direction (A, B) are

$$z_n = Ar^n + Br^{2n}.$$

The two coordinates of P_n shrink rapidly. Their projection usually does as well, except when the two terms have opposite signs and nearly cancel.

The problem in this post is taken from a geometric lemma in Hannah Cairo's paper *A Counterexample to the Mizohata–Takeuchi Conjecture*.[1] The paper uses points of this type to construct subset sums with few incidences with hyperplanes. We will only study the elementary sequence above.

There is a simple reason that rapid decay is useful. Suppose two subset sums of the P_n have the same projection. Subtracting them gives

$$\sum_{n=1}^{N} \varepsilon_n z_n = 0, \qquad \varepsilon_n \in \{-1, 0, 1\}.$$

If the first nonzero term is larger than the whole tail, this equality is impossible.

Statement

Fix $0 < \theta \le 1$. For small enough r, all but at most one index n have the property that every later term z_m is smaller than $\theta\,|z_n|$. The result is sharp. We can find A and B such that one of the terms z_n vanishes.

We use the non-sharp condition $r \le \theta/4$ because it gives a short proof.

Theorem 1. *Let $0 < \theta \le 1$ and $0 < r \le \theta/4$. For*

$$z_n = Ar^n + Br^{2n}, \qquad n = 1, \ldots, N,$$

where $A, B \in \mathbb{R}$ are not both zero, there is a set $S \subseteq \{1, \ldots, N\}$ with $|S| \le 1$ such that

$$|z_m| < \theta\,|z_n|$$

whenever $1 \le n < m \le N$ and $n \notin S$.

[1] Hannah Cairo, *A Counterexample to the Mizohata–Takeuchi Conjecture*, arXiv:2502.06137.

Proof. If $A = 0$, then

$$\frac{|z_m|}{|z_n|} = r^{2(m-n)} < \theta.$$

If $A \neq 0$ and $B/A \geq 0$, then

$$\frac{|z_m|}{|z_n|} = r^{m-n} \frac{1 + (B/A)r^m}{1 + (B/A)r^n} \leq r < \theta.$$

It remains to consider $AB < 0$.

Write $B/A = -D$ with $D > 0$, and set

$$x_n = Dr^n, \qquad f(x) = x\,|1 - x|\,.$$

Then

$$x_{n+1} = rx_n, \qquad |z_n| = \frac{|A|}{D} f(x_n).$$

The value $x = 1$ corresponds to exact cancellation.

For $0 < t < \theta$, define

$$a(t) = \frac{\theta - t}{\theta - t^2}, \qquad b(t) = \frac{\theta + t}{\theta + t^2}.$$

We claim that

$$f(tx) \geq \theta f(x) \quad \Longleftrightarrow \quad a(t) \leq x \leq b(t). \tag{1}$$

Verification of (1). The three possible ranges for x are:

$0 < x < 1$:	this is $t(1 - tx) \geq \theta(1 - x)$, equivalent to $x \geq a(t)$,
$1 < x < 1/t$:	this is $t(1 - tx) \geq \theta(x - 1)$, equivalent to $x \leq b(t)$,
$x \geq 1/t$:	this gives $t(tx - 1) \geq \theta(x - 1)$, hence $x \leq a(t) < 1/t$, impossible.

This proves the equivalence.

On the interval $0 < t \leq r \leq \theta/4$,

$$a'(t) = -\frac{\theta - 2\theta t + t^2}{(\theta - t^2)^2} < 0, \qquad b'(t) = \frac{\theta - 2\theta t - t^2}{(\theta + t^2)^2} > 0.$$

Therefore, for every $j \geq 1$,

$$[a(r^j), b(r^j)] \subseteq [a(r), b(r)].$$

Set

$$I = [a(r), b(r)].$$

If $x_n \notin I$, then (1), applied with $t = r^{m-n}$, gives

$$f(x_m) < \theta f(x_n) \qquad (m > n).$$

It remains to show that (x_n) has at most one term in I. The inequality $a(r) > rb(r)$ is equivalent to

$$r^3 + \theta r^2 + \theta r - \theta^2 < 0.$$

Since $r \le \theta/4$ and $\theta \le 1$,

$$r^3 + \theta r^2 + \theta r \le \theta^2 \left(\frac{1}{64} + \frac{1}{16} + \frac{1}{4} \right) < \theta^2.$$

Hence

$$\frac{b(r)}{a(r)} < \frac{1}{r}.$$

Suppose $n < m$ and $x_n, x_m \in I$. On the one hand,

$$\frac{x_n}{x_m} = r^{-(m-n)} \ge \frac{1}{r}.$$

On the other hand,

$$\frac{x_n}{x_m} \le \frac{b(r)}{a(r)} < \frac{1}{r},$$

a contradiction. Therefore, $|S| \le 1$, where S is the set of indices for which $x_n \in I$. □

The sharp threshold

The bound $r \le \theta/4$ was used only to simplify the proof. The exact boundary $r_\star(\theta)$ is the unique root in $(0, \theta)$ of

$$r^3 + \theta r^2 + \theta r - \theta^2 = 0.$$

The conclusion holds for every $0 < r < r_\star(\theta)$ and fails when $r \ge r_\star(\theta)$. As $\theta \to 0^+$,

$$r_\star(\theta) = \theta - 2\theta^2 + 10\theta^3 - 66\theta^4 + O(\theta^5) \sim \theta.$$

A proof of the stronger statement requires tracking I without replacing the root by the bound $\theta/4$.