Evaluation of Two Related Hypergeometric Series by Elliptic Integrals

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1 The problem and notation

We shall prove the following identity.

Theorem 1. *With the convention* $(-1)!! = 0!! = 1$*, one has*

$$\sum_{n=0}^{\infty} \frac{4n+1}{(2n-1)^2(n+1)^2} \left(\frac{(2n-1)!!}{(2n)!!} \right)^3 (-1)^n = \frac{8}{3\pi}. \tag{1}$$

We use the Pochhammer symbol

$$(a)_n = a(a+1)\cdots(a+n-1), \qquad (a)_0 = 1,$$

and the generalized hypergeometric notation

$${}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}.$$

We also write

$$\theta = z\frac{d}{dz}.$$

Thus, if $Y(z) = \sum_{n\geq 0} y_n z^n$, then

$$P(\theta)Y(z) = \sum_{n\geq 0} P(n) y_n z^n$$

for every polynomial P.

Part I

Reduction to a companion hypergeometric value

2 Hypergeometric form of the original series

Lemma 1. *Define*

$$F(z) = \sum_{n=0}^{\infty} \frac{4n+1}{(2n-1)^2(n+1)^2} \left(\frac{(2n-1)!!}{(2n)!!} \right)^3 z^n. \tag{2}$$

Then the series converges absolutely at $z = -1$, the sum in (1) is $F(-1)$, and

$$F(z) = {}_4F_3\left(\begin{matrix} \frac{5}{4}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 2, 2 \end{matrix} ; z \right). \tag{3}$$

Proof. We first use

$$\frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{4^n}\binom{2n}{n}.$$

Since $\binom{2n}{n} = O(4^n n^{-1/2})$, the nth term of (2) is $O(n^{-9/2})$. Hence the series converges absolutely at $z = -1$.

Let f_n denote the coefficient of z^n in (2). Since $f_0 = 1$, it is enough to compute the ratio of consecutive coefficients. We have

$$\frac{\binom{2n+2}{n+1}}{\binom{2n}{n}} = \frac{2(2n+1)}{n+1},$$

and therefore

$$\frac{f_{n+1}}{f_n} = \frac{(4n+5)(2n-1)^2(n+1)^2}{(4n+1)(2n+1)^2(n+2)^2} \cdot \frac{1}{64} \left(\frac{2(2n+1)}{n+1} \right)^3$$

$$= \frac{(n+\frac{5}{4})(n-\frac{1}{2})^2(n+\frac{1}{2})}{(n+\frac{1}{4})(n+2)^2(n+1)}.$$

This is exactly the coefficient ratio of the hypergeometric series in (3). $\square$

Lemma 2. *Let*

$$G(z) = {}_3F_2\left(\begin{matrix}-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \\ 2, 2\end{matrix}; z\right). \tag{4}$$

Then

$$F = (4\theta + 1)G. \tag{5}$$

Proof. For every $n \geq 0$,

$$\frac{(\frac{5}{4})_n}{(\frac{1}{4})_n} = 4n + 1.$$

Applying $4\theta + 1$ multiplies the coefficient of z^n in G by $4n + 1$, which gives (3). $\square$

Lemma 3. *Let*

$$H(z) = {}_3F_2\left(\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1\end{matrix}; z\right). \tag{6}$$

Then

$$H = (2\theta - 1)^2(\theta + 1)^2 G. \tag{7}$$

Proof. Write

$$G(z) = \sum_{n=0}^{\infty} g_n z^n, \qquad H(z) = \sum_{n=0}^{\infty} h_n z^n.$$

From the definitions,

$$\frac{h_n}{g_n} = \frac{(2)_n^2(\frac{1}{2})_n^2}{(1)_n^2(-\frac{1}{2})_n^2} = (n+1)^2(2n-1)^2.$$

The result now follows from the action of a polynomial in θ on a power series. $\square$

Lemma 4. *The functions G and H satisfy*

$$\left((1-z)\theta^3 + \frac{4+z}{2}\theta^2 + \frac{4+z}{4}\theta - \frac{z}{8}\right) G = 0 \tag{8}$$

and

$$\left(\theta^3 - z\left(\theta + \frac{1}{2}\right)^3\right) H = 0. \tag{9}$$

Proof. A function

$$y = {}_pF_{p-1}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{p-1} \end{matrix}; z\right)$$

satisfies

$$\left[\theta \prod_{j=1}^{p-1}(\theta + b_j - 1) - z \prod_{i=1}^{p}(\theta + a_i)\right] y = 0.$$

For G, this gives

$$(\theta+1)^2\theta G = z\left(\theta + \frac{1}{2}\right)\left(\theta - \frac{1}{2}\right)^2 G.$$

Expanding both sides gives (8). For H, the same formula gives (9) directly. □

3 The operator computation

Equation (7) expresses H in terms of G. We now invert this relation on the solution space of (8). This is the main computational step of the proof.

Let $\mathcal{R} = \mathbb{Q}(z)\langle\theta\rangle$ be the differential-operator ring determined by

$$\theta f = f\theta + zf', \qquad f \in \mathbb{Q}(z). \tag{10}$$

Put $\delta f = zf'$. Repeated use of (10) gives

$$(a\theta^m)(b\theta^n) = a\sum_{r=0}^{m}\binom{m}{r}\delta^r(b)\theta^{m+n-r}. \tag{11}$$

All coefficients are kept on the left of the powers of θ.

We shall use the following convention for division. Given $P, Q \in \mathcal{R}$ with $Q \neq 0$, right division means finding $S, R \in \mathcal{R}$ such that

$$P = SQ + R, \qquad \deg_\theta R < \deg_\theta Q. \tag{12}$$

Thus the divisor Q occurs on the right of the quotient S. The ordinary polynomial-division procedure still works, provided all products are computed using (11).

We now describe the extended Euclidean algorithm in the form that will be used below. Initialize

$$R_0 = A, \qquad R_1 = B, \tag{13}$$

$$U_0 = 1, \qquad U_1 = 0, \qquad V_0 = 0, \qquad V_1 = 1. \tag{14}$$

At each step, right divide R_k by R_{k+1} and write

$$R_k = S_{k+1}R_{k+1} + R_{k+2}, \qquad \deg_\theta R_{k+2} < \deg_\theta R_{k+1}. \tag{15}$$

The coefficients in the Bézout identity are updated by

$$U_{k+2} = U_k - S_{k+1}U_{k+1} \tag{16}$$

and

$$V_{k+2} = V_k - S_{k+1}V_{k+1}. \tag{17}$$

These recurrences maintain the identity

$$R_k = U_kA + V_kB \tag{18}$$

for every k. Indeed, assuming it for k and $k+1$, equations (15)–(17) give

$$\begin{aligned} R_{k+2} &= R_k - S_{k+1}R_{k+1} \\ &= (U_k - S_{k+1}U_{k+1})A + (V_k - S_{k+1}V_{k+1})B \\ &= U_{k+2}A + V_{k+2}B. \end{aligned}$$

The degrees in θ strictly decrease, so the process terminates. If the last nonzero remainder is a rational function $g(z)$, then for the corresponding operators U and V we obtain

$$UA + VB = g(z). \tag{19}$$

For the present problem, let

$$A = (1-z)\theta^3 + \frac{4+z}{2}\theta^2 + \frac{4+z}{4}\theta - \frac{z}{8}, \tag{20}$$

$$B = (2\theta-1)^2(\theta+1)^2 = 4\theta^4 + 4\theta^3 - 3\theta^2 - 2\theta + 1. \tag{21}$$

Then $AG = 0$ by (8), while $BG = H$ by (7).

Lemma 5. *Let*

$$p = z^2 + 13z + 1. \tag{22}$$

Applying the recurrence (15)–(17) *to* A *and* B *gives*

$$UA + VB = g, \tag{23}$$

where

$$\begin{aligned} U = & -\frac{16p^2}{33z(z-1)^3}\theta^3 + \frac{8(14z-19)p^2}{363z(z-1)^4}\theta^2 \\ & + \frac{20p^2}{363z(z-1)^4}\theta + \frac{2p^2}{121z(z-1)^4}, \end{aligned} \tag{24}$$

$$\begin{aligned} V = & -\frac{4p^2}{33z(z-1)^2}\theta^2 - \frac{2(19z-3)p^2}{363z(z-1)^3}\theta \\ & - \frac{(8z-13)p^2}{363(z-1)^4}, \end{aligned} \tag{25}$$

$$g = \frac{9p^2}{484(z-1)^4}. \tag{26}$$

Consequently,

$$G = CH, \tag{27}$$

where

$$C = -\frac{176(z-1)^2}{27z}\theta^2 - \frac{8(z-1)(19z-3)}{27z}\theta + \frac{52-32z}{27}. \tag{28}$$

Proof. The displayed operators are the output of the recurrence above. The identity (23) is checked in the ring $\mathcal{R}$ using the multiplication rule (11).

We now apply (23) to G. Since $AG = 0$ and $BG = H$, we get

$$VH = gG.$$

The function g is a nonzero rational function, so

$$G = g^{-1}VH.$$

Dividing the coefficients of V by g gives precisely the operator C in (28). This proves (27). The coefficients of C have apparent poles at $z = 0$, but $CH = G$ is analytic there. $\square$

We remember that $F = (4\theta + 1)G$. Therefore

$$F = (4\theta + 1)CH.$$

The operator $(4\theta+1)C$ has degree 3. We reduce it modulo the annihilating operator of H,

$$D = \theta^3 - z\left(\theta + \frac{1}{2}\right)^3. \tag{29}$$

This is another right division in the same operator ring.

Lemma 6. *Let $P = (4\theta + 1)C$. Right division of P by D gives*

$$P = QD + E, \tag{30}$$

where

$$Q = \frac{704(z-1)}{27z} \tag{31}$$

and

$$E = \frac{16(1-z^2)}{z}\theta^2 + \frac{-40z^2 - 16z + 8}{3z}\theta - \frac{8z+4}{3}. \tag{32}$$

Consequently,

$$F = EH. \tag{33}$$

Equivalently,

$$F(z) = 16z(1-z^2)H''(z) + \frac{-88z^2 - 16z + 56}{3}H'(z) - \frac{8z+4}{3}H(z). \tag{34}$$

In particular,

$$F(-1) = \frac{4}{3}\big(H(-1) - 4H'(-1)\big). \tag{35}$$

Proof. The right-division output is (30). Since $DH = 0$ by (9), we obtain

$$F = PH = QDH + EH = EH.$$

Finally, $\theta H = zH'$ and $\theta^2 H = zH' + z^2 H''$. Substitution into (32) gives (34). Setting $z = -1$ gives (35). □

Part II

Evaluation of the companion hypergeometric value

The first six lemmas have reduced the original sum to a value built from H and H'. Before evaluating it, we record that this quantity is itself a natural hypergeometric series.

Lemma 7 (The companion series and Bauer's series)**.** *Define*

$$\mathcal{B} = H(-1) - 4H'(-1). \tag{36}$$

Then

$$\mathcal{B} = {}_4F_3\left(\begin{matrix}\frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1\end{matrix}; -1\right) \tag{37}$$

$$= \sum_{n=0}^{\infty} (4n+1) \left(\frac{(2n-1)!!}{(2n)!!}\right)^3 (-1)^n. \tag{38}$$

Consequently,

$$F(-1) = \frac{4}{3}\mathcal{B}. \tag{39}$$

Thus the remainder of the proof simultaneously evaluates the original series and the companion series (38)*, commonly known as Bauer's series.*

Proof. Write

$$H(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad c_n = \frac{(\frac{1}{2})_n^3}{(n!)^3} = \left(\frac{(2n-1)!!}{(2n)!!}\right)^3.$$

The sequence nc_n tends to 0 and is decreasing for $n \geq 1$, because

$$\frac{(n+1)c_{n+1}}{nc_n} = \frac{(n+\frac{1}{2})^3}{n(n+1)^2} < 1.$$

Hence the differentiated series converges uniformly on the real interval $[-1, 0]$ by the uniform alternating test, and termwise differentiation is valid there. Therefore

$$H(-1) - 4H'(-1) = \sum_{n=0}^{\infty} c_n (-1)^n - 4 \sum_{n=1}^{\infty} nc_n (-1)^{n-1}$$

$$= \sum_{n=0}^{\infty} (4n+1) c_n (-1)^n,$$

which proves (38). Moreover,

$$\frac{(\frac{5}{4})_n}{(\frac{1}{4})_n} = 4n+1,$$

so the same coefficient identity gives (37). Finally, (39) is exactly (35). $\square$

4 Hypergeometric transformations

Lemma 8 (Clausen's identity)**.** *Let*

$$I(z) = {}_2F_1\left(\begin{matrix} \frac{1}{4}, \frac{1}{4} \\ 1 \end{matrix}; z\right). \tag{40}$$

Then

$$H(z) = I(z)^2. \tag{41}$$

In particular,

$$\mathcal{B} = I(-1)\big(I(-1) - 8I'(-1)\big). \tag{42}$$

Proof. Clausen's identity in the form

$${}_2F_1\left(\begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}; z\right)^2 = {}_3F_2\left(\begin{matrix} 2a, 2b, a+b \\ 2a+2b, a+b+\frac{1}{2} \end{matrix}; z\right)$$

with $a = b = \frac{1}{4}$ gives (41). The identity is first valid for $|z| < 1$ and extends to $z = -1$ by continuity. Differentiating $H = I^2$ and using the definition (36) gives (42). $\square$

Define

$$T_K(z) = {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; z\right), \qquad T_E(z) = {}_2F_1\left(\begin{matrix} -\frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; z\right). \tag{43}$$

Lemma 9 (Quadratic transformation)**.** *One has*

$$I(4z(1-z)) = T_K(z). \tag{44}$$

Let
$$z_0 = \frac{1-\sqrt{2}}{2}. \tag{45}$$
Then
$$\mathcal{B} = T_K(z_0)\big(T_K(z_0) - \sqrt{2}\,T_K'(z_0)\big). \tag{46}$$

Proof. Equation (44) is Kummer's quadratic transformation. Since
$$4z_0(1-z_0) = -1, \qquad 4-8z_0 = 4\sqrt{2},$$
we obtain
$$I(-1) = T_K(z_0), \qquad I'(-1) = \frac{T_K'(z_0)}{4\sqrt{2}}.$$
Substitution into (42) gives (46). □

Lemma 10 (Pfaff's transformation)**.** *Put*
$$z_1 = \frac{z_0}{z_0-1} = 3-2\sqrt{2}. \tag{47}$$
Then
$$\mathcal{B} = (6-4\sqrt{2})T_K(z_1)^2 + (80-56\sqrt{2})T_K(z_1)T_K'(z_1). \tag{48}$$

Proof. Pfaff's transformation gives
$$T_K(z) = (1-z)^{-1/2}T_K\left(\frac{z}{z-1}\right). \tag{49}$$
Let $w = z/(z-1)$. Differentiating (49),
$$T_K'(z) = \frac{1}{2}(1-z)^{-3/2}T_K(w) - (1-z)^{-5/2}T_K'(w).$$
At $z = z_0$, we have $w = z_1$ and $1-z_0 = (1+\sqrt{2})/2$. Substituting these values into (46) and simplifying gives
$$T_K(z_0)\big(T_K(z_0)-\sqrt{2}T_K'(z_0)\big) = (6-4\sqrt{2})T_K(z_1)^2+(80-56\sqrt{2})T_K(z_1)T_K'(z_1),$$
which proves the result. □

Lemma 11. *For $z \neq 0, 1$,*
$$T_K'(z) = \frac{T_E(z)-(1-z)T_K(z)}{2z(1-z)}. \tag{50}$$
Consequently,
$$\mathcal{B} = -2T_K(z_1)^2 + 2\sqrt{2}\,T_K(z_1)T_E(z_1). \tag{51}$$

Proof. We first prove

$$T_E(z) = (1-z)(2\theta+1)T_K(z). \tag{52}$$

Let $k_n = (\frac{1}{2})_n^2/(n!)^2$ be the coefficient of z^n in T_K. For $n \geq 1$, the coefficient of z^n on the right side of (52) is

$$(2n+1)k_n - (2n-1)k_{n-1} = -\frac{k_n}{2n-1}.$$

On the other hand,

$$\frac{(-\frac{1}{2})_n(\frac{1}{2})_n}{(n!)^2} = -\frac{k_n}{2n-1},$$

and the constant terms are also equal. Thus (52) holds. Since

$$(1-z)(2\theta+1)T_K = (1-z)T_K + 2z(1-z)T_K',$$

we obtain (50).

Now

$$\frac{80-56\sqrt{2}}{2z_1(1-z_1)} = 2\sqrt{2}.$$

Substituting (50) into (48) gives

$$\begin{aligned}\mathcal{B} &= (6-4\sqrt{2})T_K(z_1)^2 + 2\sqrt{2}\,T_K(z_1)T_E(z_1)\\ &\quad - 2\sqrt{2}(1-z_1)T_K(z_1)^2.\end{aligned}$$

Since $2\sqrt{2}(1-z_1) = 8-4\sqrt{2}$, the coefficient of $T_K(z_1)^2$ is -2, proving (51). $\square$

5 Elliptic integral evaluation

The complete elliptic integrals of the first and second kind are

$$K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}, \qquad E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2 t}\,dt. \tag{53}$$

Their hypergeometric forms are

$$K(k) = \frac{\pi}{2}T_K(k^2), \qquad E(k) = \frac{\pi}{2}T_E(k^2). \tag{54}$$

Lemma 12. *Let*

$$k = \sqrt{2} - 1. \tag{55}$$

Then $k^2 = z_1$ *and*

$$\mathcal{B} = \frac{4}{\pi^2}\left(-2K(k)^2 + 2\sqrt{2}\,K(k)E(k)\right). \tag{56}$$

Proof. Since $(\sqrt{2}-1)^2 = 3 - 2\sqrt{2} = z_1$, the result follows immediately from (51) and (54). □

Lemma 13. *For* $k = \sqrt{2} - 1$,

$$2\sqrt{2}\,K(k)E(k) - 2K(k)^2 = \frac{\pi}{2}. \tag{57}$$

Proof. Let $k' = \sqrt{1-k^2}$. Legendre's relation is

$$E(k)K(k') + E(k')K(k) - K(k)K(k') = \frac{\pi}{2}. \tag{58}$$

We also use the ascending Landen formulas. If

$$k_2 = \frac{2\sqrt{k}}{1+k},$$

then

$$K(k_2) = (1+k)K(k), \qquad E(k_2) = \frac{2E(k) - k'^2K(k)}{1+k}. \tag{59}$$

For $k = \sqrt{2} - 1$, we have

$$1 - k^2 = 2k, \qquad (1+k)^2 = 2.$$

It follows that

$$k_2^2 = \frac{4k}{(1+k)^2} = 2k = 1 - k^2 = k'^2,$$

so $k_2 = k'$. Substituting (59) into (58) gives

$$\begin{aligned}\left(1 + k + \frac{2}{1+k}\right) E(k)K(k)&\\ - \left(\frac{k'^2}{1+k} + 1 + k\right) K(k)^2 &= \frac{\pi}{2}.\end{aligned}$$

Now $1 + k = \sqrt{2}$ and $k'^2 = 2(\sqrt{2} - 1)$. Hence

$$1 + k + \frac{2}{1+k} = 2\sqrt{2}, \qquad \frac{k'^2}{1+k} + 1 + k = 2.$$

This proves (57). □

Proof of the theorem. Combining Lemmas 12 and 13, we obtain

$$\mathcal{B} = \frac{4}{\pi^2} \cdot \frac{\pi}{2} = \frac{2}{\pi}.$$

Equation (39) now gives

$$F(-1) = \frac{4}{3}\mathcal{B} = \frac{8}{3\pi}.$$

By Lemma 1, $F(-1)$ is the series in (1). This completes the proof. $\square$

Corollary 2 (Bauer's formula)**.** *One has*

$$ {}_4F_3\left(\begin{matrix} \frac{5}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{4}, 1, 1 \end{matrix}; -1\right) = \sum_{n=0}^{\infty} (4n+1) \left(\frac{(2n-1)!!}{(2n)!!}\right)^3 (-1)^n = \frac{2}{\pi}. \tag{60}$$

Proof. This is the value of $\mathcal{B}$ obtained in the proof of the theorem, together with Lemma 7. $\square$